ON TVERBERG'S TRANSVERSAL CONJECTURE AND ANALOGUES OF NONEMBEDDABILITY THEOREMS FOR TRANSVERSALS

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Abstract. In this paper we prove a special case of the transversal conjecture of Tverberg and Vrećica. We consider the case when the numbers of parts r_i in this conjecture are powers of the same prime. We also prove some results on common transversals that generalize the classical nonembeddability theorems.

We also prove an analogue of colored Tverberg's theorem by Živaljević and Vrećica. Instead of multicolor simplices with common point it gives multicolor simplices with common m-transversal.

1. Tverberg's transversal conjecture

In this paper we prove a special case of the transversal conjecture of Tverberg and Vrećica.

Conjecture 1. *Let $0 \le m \le d-1$ and let $S_0, S_1, \dots, S_m$ be $m+1$ finite sets in $\mathbb{R}^d$. Let $|S_i| = (r_i-1)(d-m+1)+1$. Then every set S_i can be partitioned into r_i parts $S_{i1}, S_{i2}, \dots, S_{ir_i}$ so that all the sets* $\operatorname{conv} S_{ij}$ *can be met by a single m-flat.*

This conjecture was formulated by H. Tverberg on the 1989 Symposium on Combinatorics and Geometry in Stockholm. In print it was first formulated by H. Tverberg and S. Vrećica in [7], where a special case of this conjecture was proved.

In the papers [11, 12] R. Živaljević and S. Vrećica established the case of this conjecture when all r_i are equal to the same prime p, if p is odd then d and m were also required to be odd.

Here we prove the theorem that generalizes the results of R. Živaljević and S. Vrećica to prime powers and show that the condition that d is odd is not necessary. Besides, our proof is quite short because of using the multiplicative rule for the Euler class.

We prove that this conjecture is true when the numbers r_i are powers of the same prime $r_i = p^{k_i}$, and for odd p, we also need $d-m$ to be even.

Similarly to what was done by R. Živaljević and S. Vrećica, we prove a more general topological version of this conjecture.

Theorem 1. *Let $0 \le m \le d-1$, let r_i $(i = 0, \dots, m)$ be powers of the same prime $r_i = p^{k_i}$. If p is odd, let $d-m$ be even.*

Let for each $i = 0, \dots, m$ f_i map continuously an $(r_i-1)(d-m+1)$-dimensional simplex $\Delta_i = \Delta^{(r_i-1)(d-m+1)}$ to $\mathbb{R}^d$.

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Then every simplex Δ_i has r_i points $x_{i1}, x_{i2}, \dots, x_{ir_i} \in \Delta_i$ with pairwise disjoint supports so that all the points $f_i(x_{ij})$ are contained in a single m-flat.

2. Transversal analogues of nonembeddability theorems

In this paper we generalize some results of [11] that give transversal analogues of nonembeddability theorems. We have to remind some basic concepts in equivariant topology, for detailed explanation see [13], chapters III and IV.

Let us take the classifying G-space EG and define G-equivariant cohomology of a G-space X as

$$H^*_G(X,K) = H^*((X \times EG)/G, K).$$

Now consider the case $G = Z_2$. The cohomology ring of the one-point space $\Lambda = H^*_{Z_2}(\mathrm{pt}, Z_2)$ can be represented as $\Lambda = Z_2[u]$, where $\dim u = 1$. If Z_2 acts on $\mathbb{R}^n$ by sending x to $-x$, then the equivariant Euler class of this representation is

$$e(\mathbb{R}^n) = u^n \in H^n_{Z_2}(\mathrm{pt}, Z_2).$$

For any space Y with Z_2 action the equivariant map $Y \to \mathrm{pt}$ gives the canonical map $\Lambda \to H^*_{Z_2}(Y, Z_2)$. If this does not lead to a confusion, we denote the images of u^k under this map by u^k.

Definition. Let Y be a Z_2-space. The maximal n such that $u^n \neq 0 \in H^n_{Z_2}(Y, Z_2)$ (it may be $+\infty$) is called the *homological index* of Y. We shall denote it $\operatorname{hind} Y$.

For any topological space we denote $X^2_\Delta = X \times X \setminus \Delta(X)$ the deleted product of X, i.e. the square of X without the diagonal.

For a simplicial complex K it is useful to define the deleted join K^{*2}_Δ in the following way. Let the vertex set of K be V. Then the vertex set of K^{*2}_Δ is $V_1 \sqcup V_2$, the disjoint union of two copies of V. The simplices of K^{*2}_Δ are pairs of simplices of K $\sigma_1 \subseteq V_1$, $\sigma_2 \subseteq V_2$ such that considered as subsets of V they have $\sigma_1 \cap \sigma_2 = \emptyset$.

Deleted squares and joins have natural Z_2 action by permutation. As it is discussed in [15], the simplest way to prove that some topological space X cannot be embedded into $\mathbb{R}^n$ is to show that $\operatorname{hind} X^2_\Delta \geq n$. Similarly, for a simplicial complex K it is sufficient to show that $\operatorname{hind} K^{*2}_\Delta \geq n+1$.

Now we can formulate the transversal analogues of nonembeddability theorems.

Theorem 2. *Let $X_0, X_1, \dots, X_m$ be topological spaces such that for any $i = 0, \dots, m$ $\operatorname{hind}(X_i)^2_\Delta \geq d - m$. Suppose for each $i = 0, \dots, m$ we have a continuous map $f_i : X_i \to \mathbb{R}^d$. Then there exist $m+1$ pairs of distinct points $x_i, y_i \in X_i$ such that their images $\{f_i(x_i), f_i(y_i)\}_{i=0}^m$ are met by some m-flat in $\mathbb{R}^d$.*

Theorem 3. *Let $K_0, K_1, \dots, K_m$ be simplicial complexes such that for any $i = 0, \dots, m$ $\operatorname{hind}(K_i)^{*2}_\Delta \geq d - m + 1$. Suppose for each $i = 0, \dots, m$ we have a continuous map $f_i : K_i \to \mathbb{R}^d$. Then there exist $m+1$ pairs of points $x_i, y_i \in K_i$ with disjoint support simplices such that their images $\{f_i(x_i), f_i(y_i)\}_{i=0}^m$ are met by some m-flat in $\mathbb{R}^d$.*

For example, if we consider the d-skeleton of $2d+2$-dimensional simplex $K = \Delta^d_{2d+2}$, then $\operatorname{hind}(K^{*2}_\Delta) = 2d+1$. This is shown in [15] for another definition of index, but in fact the reasoning is the same for homological index.

Using this fact we can formulate the following corollary that generalizes the Van Kampen-Flores theorem from [1, 2], which is the case $m = 0$ of the following statement.

Corollary 4. *If we have $m+1$ continuous maps $f_i : \Delta^d_{2d+2} \to \mathbb{R}^{2d+m}$, then there exist $m+1$ pairs of points $x_i, y_i \in \Delta^d_{2d+2}$ with disjoint supports in each pair such that their images $\{f_i(x_i), f_i(y_i)\}_{i=0}^m$ are met by some m-flat in $\mathbb{R}^{2d+m}$.*

3. Colored version of Tverberg's transversal conjecture

Denote $[n] = \{1, 2, \dots, n\}$

Definition. Let set S be colored into n colors (i.e. mapped to $[n]$). Call a nonempty subset $\sigma \subseteq S$ multicolored if each color occurs in σ no more than once.

Theorem 5. *Let $0 \le m \le d-1$ and let $S_0, S_1, \dots, S_m$ be $m+1$ finite sets in $\mathbb{R}^d$.*

Let r_i $(i = 0, \dots, m)$ be powers of the same prime p. Let k be the number of colors and either $k = d-m+1$ or $k < d-m+1$ and for each $i = 0, \dots, m$ $r_i \le \dfrac{d-m}{d-m+1-k}$. If $p \ne 2$ we require that $d-m$ is even. Let $t_i = 2r_i - 1$.

Let $|S_i| = t_i k$ and let each S_i be colored in k colors so that each color is used in S_i t_i times.

Then we can find r_i disjoint multicolored subsets for each i

$$P_{i1}, P_{i2}, \dots, P_{ir_i} \subseteq S_i$$

so that all the sets $\operatorname{conv} P_{ij}$ *$(i = 0, \dots, m,\ j = 1, \dots, r_i)$ can be met by a single m-flat.*

We also formulate another version of this theorem

Theorem 6. *Theorem 5 is also true when $k = d+1-m$ and for those of r_i that equal 2 we take $t_i = 2$ instead of 3.*

Theorems 5 and 6 are straightforward generalizations of colored Tverberg's theorem from [6, 8, 10]. They generalize colored Tverberg's theorem in the same way as results from [11, 12] and Theorem 1 generalize the non-colored Tverberg's theorem to m-transversals.

It will be seen from the proof that the sets S_i can be replaced by continuous maps of simplicial complexes K_i (defined below) to $\mathbb{R}^d$ and $\operatorname{conv} P_{ij}$ can be replaced by the images of pairwise disjoint simplices of K_i. That means that we prove some stronger topological versions of Theorems 5 and 6 in fact.

Similar to the colored Tverberg's conjecture, it is natural to ask whether the number $t = 2r-1$ in Theorem 5 can be made less, or even whether it can be $t = r$ when $r \ne 2$. Colored Tverberg's conjecture for $t = r \ne 2$ is only proved in [5] for $d = 2$ by a non-topological method, so making t less in Theorem 5 seems to be quite a hard problem.

4. Constructions with vector spaces

We use constructions from the topological proof of Tverberg's theorem [3] as they are used in the book [15]. We use them for the case of prime powers, following the ideas from [9].

Consider a vector space V and an integer $n > 0$. Let us make a definition.

Definition. Take $\mathbb{R}^n$ with coordinates $(t_1, \dots, t_n)$ and consider the hyperplane A_n given by the equation

$$t_1 + t_2 + \dots + t_n = 1.$$

If we want to treat A_n as a vector space, we put its origin to $(1/n, 1/n, \dots, 1/n)$.

Definition. For a vector space V and an integer $n > 0$ put

$$J^n_A(V) = nV \oplus A_n,$$

here nV is the direct sum of n copies of V. The space V can be embedded into $J^n_A(V)$ with the map

$$v \mapsto v \oplus v \oplus \dots \oplus v \oplus (1/n, \dots, 1/n),$$

and this gives the orthogonal decomposition

$$J^n_A(V) = V \oplus D^n_A(V).$$

In fact it can be easily seen that $J^n_A(V)$ is the affine hull of the n-fold join $V * V * \dots * V$.

Action of any group G on the index set $[n]$ gives an action on nV by permutations of the summands, it also gives an action on A_n. The summand V in the decomposition $J^n_A(V) = V \oplus D^n_A(V)$ is stable under this action. If G acts transitively on $[n]$, then its representation on $D^n_A(V)$ has no trivial summands.

In the sequel we consider $G = (Z_p)^k$ (p is a prime) and choose some bijection between $[n]$ ($n = p^k$) and G. Thus G acts on $[n]$ by shifts.

In this case the representation $D^n_A(V)$ has no trivial summands and (see [13], chapter IV §1 for description of $H^*_G(\mathrm{pt}, Z_p)$) its Euler class $e(D^n_A(V)) \neq 0 \in H^*_G(\mathrm{pt}, Z_p)$.

5. Topological constructions

Let $r = p^k$, p be a prime, $G = (Z_p)^k$. Consider the N-fold join $EG_N = [r] * [r] * \dots * [r]$. It is known that it is $N-1$-dimensional and $N-2$-connected, see the book [15] for very clear explanation of these matters.

The action of G on $[r]$ induces the free action of G on EG_N and the canonical map

$$H^*_G(\mathrm{pt}, Z_p) \to H^*_G(EG_N, Z_p)$$

is injective in dimesions $\leq N-1$, similar to what is done in [9].

In the proof of Theorem 1 it is crucial that the simplicial complex EG_N is G-equivariantly isomorphic to the r-fold deleted join of $N-1$-dimensional simplex, i.e.

$$EG_N = (\Delta^{N-1})^{*r}_{\Delta}.$$

The space of deleted join is the subset of ordinary join $(\Delta^{N-1})^{*r}$ consisting of affine combinations of r points of Δ^{N-1}

$$c_1 x_1 \oplus c_2 x_2 \oplus \dots \oplus c_r x_r$$

with pairwise disjoint supports.

In the proof of Theorem 5 we have another configuration space. Consider the following simplicial complex K. Let its vertex set be $S = [k] \times [t]$ and let the first factor $[k]$ denote the color of the vertex. Let simplices of K be all multicolored subsets $\sigma \subseteq S$.

Let us describe the r-fold deleted join $L = K_{\Delta}^{*r}$. Its vertex set is $[r] \times S$, every simplex $\sigma \subseteq L$ can considered as a union

$$\sigma = 1 \times \sigma_1 \cup 2 \times \sigma_2 \cup \ldots \cup r \times \sigma_r,$$

where $\{\sigma_1, \ldots, \sigma_r\}$ are disjoint simplices of K. In the papers [6, 8] it is shown that the complex L is $rk-2$-connected.

The action of G on $[r]$ induces the free action of G on L and the canonical map

$$H_G^*(\mathrm{pt}, Z_p) \to H_G^*(L, Z_p)$$

is injective in dimesions $\leq rk-1$, this fact can be deduced from the Serre spectral sequence of the fibration $(EG \times L)/G \to BG$, see [13], chapter III §1.

Next we remind Künneth's formula (see [14]) for cohomology with coefficients in some field F

$$H^n(X \times Y, F) = \bigoplus_{k=0}^{n} H^k(X, F) \otimes H^{n-k}(Y, F),$$

where the map from $H^k(X, F) \times H^{n-k}(Y, F)$ to $H^n(X \times Y, F)$ is given by $\times$-product.

We formulate the following simple statement as a separate lemma, needed in the proofs below.

Lemma 7. *If the space Y is connected then some inclusion $i : X = X \times \{y_0\} \to X \times Y$ gives the map*

$$i^* : \bigoplus_{k=0}^{n} H^k(X, F) \otimes H^{n-k}(Y, F) \to H^n(X, F),$$

which is the projection onto the n-th summand of $\bigoplus_{k=0}^{n} H^k(X, F) \otimes H^{n-k}(Y, F)$, having in mind that $H^0(Y, F) = F$.

The definition of equivariant cohomology immediately gives for G_1-space X and G_2-space Y equivariant Künneth's formula:

$$H^n_{G_1 \times G_2}(X \times Y, F) = \bigoplus_{k=0}^{n} H^k_{G_1}(X, F) \otimes H^{n-k}_{G_2}(Y, F).$$

If we consider two (equivariant) oriented vector bundles $\xi \to X$ and $\eta \to Y$, then the space $\xi \times \eta$ is a vector bundle over $X \times Y$ and we have the multiplicative rule for the Euler class

$$e(\xi \times \eta) = e(\xi) \times e(\eta).$$

When the bundles are over the same space and we take their fiber-wise direct sum we have (we use simple multiplicative notation for the cup product)

$$e(\xi \oplus \eta) = e(\xi)e(\eta).$$

6. Lemma about the Grassmann variety

Consider the canonical bundle over the Grassmann variety $\gamma \to G_d^{d-m}$. In the case $p = 2$ we consider the variety of non-oriented $d-m$-subspaces, and for odd p we consider the variety of oriented subspaces. If we have to distinguish between these cases we denote the oriented Grassmann variety by G_d^{+d-m}

Lemma 8. *For the Euler class $e(\gamma)$ modulo p the following holds*

$$e(\gamma)^m \neq 0 \in H^{m(d-m)}(G_d^{d-m}, Z_p),$$

if $p = 2$ or $d-m$ is even.

The case of this lemma for $p = 2$ is proved in [4]. The case for p odd and d odd is proved in [11].

Proof. Here we give the proof for the case of odd p and the oriented Grassmann variety G_d^{+d-m}. Denote $l = d - m$.

We use the method of test sections. That means that to prove that an N-dimensional oriented vector bundle η over an N-dimensional oriented manifold M has non-zero Euler class we should make the following. Take some section of this bundle and consider its zero points. At each zero point consider some its vicinity with properly oriented coordinates $x_1, \dots, x_N$, the bundle here can be considered trivial with properly oriented coordinates $u_1, \dots, u_N$. The section is locally given by

$$\begin{aligned} u_1 &= h_1(x_1, \dots, x_N) \\ &\dots \\ u_N &= h_N(x_1, \dots, x_N), \end{aligned}$$

and the index of this zero point is $\operatorname{sgn}\det\left(\dfrac{\partial h_i}{\partial x_j}\right)$. If all zero points of the section are transversal (i.e. their indices are non-zero) the Poincaré duality (see [14]) shows that $e(\eta) = sw(M)$, where s is the sum of indices of all zero points and $w(M)$ is the fundamental N-dimensional class of M.

Now we have ml-dimensional manifold G_d^{+l} and ml-dimensional bundle $\gamma \oplus \dots \oplus \gamma = m\gamma$ (m-fold direct sum of γ).

Take m orthonormal vectors $f_1, f_2, \dots, f_m \in \mathbb{R}^d$ and consider sections of γ that are formed by orthogonal projections of f_i to the corresponding l-subspace $L \in G_d^{+l}$.

We denote these sections by $f_1, \dots, f_m$ too. Together they form a section $f = (f_1, f_2, \dots, f_m)$ of $m\gamma$.

As it is easily seen, f has two zero points in G_d^{+l}. They correspond to the l-subspace L that is orthogonal to vectors f_i with two possible orientations.

Choose the orthonormal base $e_1, \dots, e_l \in L$. Consider the case when $e_1 \wedge \dots \wedge e_l \wedge f_1 \wedge \dots \wedge f_m$ is positive. The elements of G_d^{+l} close to L can be uniquely described by the base

of form

$$\begin{aligned} e'_1 &= e_1 + x_{11}f_1 + \cdots + x_{1m}f_m \\ &\cdots \\ e'_l &= e_l + x_{l1}f_1 + \cdots + x_{lm}f_m, \end{aligned}$$

and the orientation of G_d^{+l} is given by (row-wise product)

$$\omega = dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{1m} \wedge \ldots \wedge dx_{l1} \wedge \ldots \wedge dx_{lm}.$$

The section f of γ locally has the form as a map $G_d^{+l} \to \mathbb{R}^{lm}$ (column-wise order)

$$(x_{11}, x_{21}, \ldots, x_{l1}) \oplus \cdots \oplus (x_{1m}, x_{2m}, \ldots, x_{lm}).$$

We see that this is a transversal zero point and its index equals the sign of permutation that is induced by the change of row-wise to column-wise order, denote it by $I = \pm 1$.

Now if we consider the base of L with different orientation, we have the same index for section $f' = (f_2, f_1, \ldots, f_m)$. Then transposing f_1 and f_2 does not change the index, since l is even. Thus the second point has the same index I.

Finally we have that $e(m\gamma) = e(\gamma)^m = I[G_d^{+l}] = \pm 2[G_d^{+l}]$, which is not zero modulo odd primes. □

7. Proof of Theorem 1

Put for all $i = 0, \ldots, m$ $G_i = (Z_p)^{k_i}$, $N_i = (r_i - 1)(d - m + 1) + 1$.

Denote $l = d - m$. The Grassmann variety will be oriented for odd p.

In this theorem we need r_i-tuples of points in each Δ_i with pairwise disjoint supports, so it is natural to consider the r_i-fold deleted joins $EG_{N_i} = (\Delta^{N_i-1})^{*r_i}_{\Delta}$ as proper configuration spaces, as it is frequently done in [15].

For any l-dimensional linear subspace $V \subseteq \mathbb{R}^d$ consider the deleted join map

$$h_{i,V} : EG_{N_i} \to J^{r_i}_A(V),$$

note that the space to the right contains r_i-fold join of V.

The maps $h_{i,V}$ are equivariant under the actions of G_i. Take the cartesian product of these maps to get the $G = G_0 \times \cdots \times G_m$-equivariant map

$$h_{0,V} \times \cdots \times h_{m,V} : EG_{N_0} \times \cdots \times EG_{N_m} \to J^{r_0}_A(V) \oplus \cdots \oplus J^{r_m}_A(V).$$

Now considering the dependence of this map on $V \in G_d^l$ we get an equivariant section of the bundle

$$\xi = J^{r_0}_A(\gamma) \oplus \cdots \oplus J^{r_m}_A(\gamma) \to EG_{N_0} \times \cdots \times EG_{N_m} \times G_d^l.$$

Now consider the decomposition

$$\xi = D^{r_0}_A(\gamma) \oplus \cdots \oplus D^{r_m}_A(\gamma) \oplus (m+1)\gamma$$

and with the diagonal embedding $\gamma \to (m+1)\gamma$ we also have the decomposition $(m+1)\gamma = \gamma_\Delta \oplus \eta$. From here on we choose some isomorphism between η and $m\gamma$, in matrix form it may

be given by some orthogonal basis in the orthogonal complement of the vector $(1,\dots,1)$ in $\mathbb{R}^{m+1}$.

Thus denote

$$\zeta = D^{r_0}_A(\gamma)\oplus\dots\oplus D^{r_m}_A(\gamma)\oplus\eta\to EG_{N_0}\times\dots\times EG_{N_m}\times G^l_d.$$

Note that if the section constructed above in projection to ζ has a zero, then we are done. Indeed, in this case we have some V and $m+1$ points $y_i\in EG_{N_i}$. Now remember that every EG_{N_i} is a deleted join, that means that for all $i=0,\dots,m$ y_i is an affine combination

$$y_i = c_{i1}x_{i1}\oplus\dots\oplus c_{ir_i}x_{ir_i}$$

of points in Δ_i with disjoint supports. The condition that the section has zero means that for all i and j $c_{ij}=\dfrac{1}{r_i}$ and the projections of $f_i(x_{ij})$ to V coincide and give a point v. Taking the affine m-subspace of $\mathbb{R}^d$ orthogonal to V and passing through v we get what we need.

Finally we have to show that the equivariant Euler class

$$e(\zeta)\in H^*_G(EG_{N_0}\times\dots\times EG_{N_m}\times G^l_d, Z_p)$$

is nonzero. Note that by Lemma 8 we have $e(\eta)\neq 0\in H^*(G^l_d,Z_p)$. Also note that any embedding with fixed $V\in G^l_d$

$$g: EG_{N_0}\times\dots\times EG_{N_m}\to EG_{N_0}\times\dots\times EG_{N_m}\times G^l_d$$

induces the bundle

$$\beta = g^*(D^{r_0}_A(\gamma)\oplus\dots\oplus D^{r_m}_A(\gamma))\to EG_{N_0}\times\dots\times EG_{N_m},$$

which is the cartesian product of G_i-bundles over EG_{N_i}, arising from representations of G_i in $D^{r_i}_A(V)$. Since $\dim D^{r_i}_A(V)=N_i-1$, the construction of EG_{N_i} implies that each of these bundles has nonzero Euler class in $H^*(EG_{N_i},Z_p)$ and by the multiplicative rule and Künneth's formula

$$e(\beta)\neq 0\in H^*_G(EG_{N_0}\times\dots\times EG_{N_m},Z_p).$$

Using Künneth's formula for $(EG_{N_0}\times\dots\times EG_{N_m})/G\times G^l_d$ and Lemma 7 we see that

$$e(D^{r_0}_A(\gamma)\oplus\dots\oplus D^{r_m}_A(\gamma)) = e(\beta)\times 1+\sum u\times v,$$

where $u\in H^*_G(EG_{N_0}\times\dots\times EG_{N_m},Z_p)$, $v\in H^*(G^l_d,Z_p)$ and all v have $\dim v>0$.

Now by the multiplicative rule

$$e(\zeta)=e(D^{r_0}_A(\gamma)\oplus\dots\oplus D^{r_m}_A(\gamma))e(\eta)=e(\beta)\times e(\eta)+\sum u\times ve(\eta)\neq 0$$

in the cohomology $H^*_G(EG_{N_0}\times\dots\times EG_{N_m}\times G^l_d,Z_p)$.

8. Proof of Theorems 2 and 3

In the following proofs we also denote $l = d - m$.

Proof of Theorem 2. Consider some l-subspace V of $\mathbb{R}^d$ and denote orthogonal projection to V by π_V. For any $i = 0, \dots, m$ consider the map $s_{i,V} : (X_i)^2_\Delta \to V$ given by

$$s_{i,V}(x, y) = \pi_V(f_i(x)) - \pi_V(f_i(y)).$$

This is Z_2-equivariant map if Z_2 acts on V antipodally. These maps for $i = 0, \dots, m$ form a map

$$s_V : (X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \to V \oplus \cdots \oplus V = (m+1)V.$$

Also consider the maps $t_{i,V} : (X_i)^2_\Delta \to V$ given by

$$t_{i,V}(x, y) = \pi_V(f_i(x)) + \pi_V(f_i(y)),$$

the product of these maps gives a map $t_V : (X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \to (m+1)V$. The space $(m+1)V$ can be decomposed into its diagonal and its orthogonal complement $(m+1)V = V_\Delta \oplus W$. So we consider the composition of t_V with the projection to W and get the map $u_V : (X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \to W$.

Then we take the map

$$s_V \oplus u_V : (X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \to (m+1)V \oplus W,$$

this map can be considered as a section of the vector bundle $\xi \oplus \eta = (m+1)\gamma \oplus m\gamma$ over $(X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \times G^l_d$. Here we again choose some isomorphism between the bundle η arising from spaces W and $m\gamma$.

The constructed section is equivariant w.r.t. action of $G = (Z_2)^{m+1}$ on $\xi \oplus \eta = (m+1)\gamma \oplus m\gamma$ by antipodal maps on the former $m+1$ summands.

It is easy to see that if this section has a zero, then the projections of corresponding points $\{f_i(x_i), f_i(y_i)\}_{i=0}^m$ to V coincide and the theorem is proved.

All we have to do is to prove that the Euler class of this bundle is nonzero modulo 2. By Lemma 8

$$e(\eta) = e(\gamma)^m \neq 0 \in H^{m(d-m)}(G^l_d, Z_2).$$

Taking some fixed $V \in G^l_d$ we have a map

$$(X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \to (X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \times G^l_d$$

that induces a bundle ξ' over $(X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta$. By Künneth's formula, multiplicative rule for the Euler class, and the index property of $(X_i)^2_\Delta$ we have

$$e(\xi') = u^l \times \cdots \times u^l \neq 0 \in H^{l(m+1)}_G((X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta, Z_2).$$

This means that by Künneth's formula the Euler class $e(\xi)$ in $H^*_G((X_0)^2_\Delta \times \cdots \times (X_m)^2_\Delta \times G^l_d, Z_2)$ has the form $e(\xi) = e(\xi') \times 1 + \sum a \times b$ (by Lemma 7), where $\dim b > 1$ always.

Then by the multiplicative rule we see that $e(\xi \oplus \eta) = e(\xi') \times e(\eta) + \sum a \times be(\eta) \neq 0$. $\square$

Proof of Theorem 3. The proof is similar to the previous one, so we only show the differences.

Instead of maps $s_{i,V}$ and $t_{i,V}$ we consider the map $r_{i,V} : (K_i)^{*2}_\Delta \to V \oplus V \oplus L$, where L is one-dimensional linear space. This map is given by

$$r_{i,V}(tx \oplus (1-t)y) = t\pi_V(f_i(x)) \oplus (1-t)\pi_V(f_i(y)) \oplus (t-1/2).$$

This map is obviously equivariant w.r.t. Z_2-action on $(K_i)^{*2}_\Delta$ and its action on $V \oplus V \oplus L$ that permutes two summands V and acts antipodally on L.

The considered space $V \oplus V \oplus L$ may be decomposed into $U_i \oplus W_i$, where Z_2 acts antipodally on $l+1$-dimensional space U_i and trivially on l-dimensional space W_i.

We sum up the maps r_i to have the map

$$r_V : (K_0)^{*2}_\Delta \times \cdots \times (K_m)^{*2}_\Delta \to (U_0 \oplus \cdots \oplus U_m) \oplus (W_0 \oplus \cdots \oplus W_m) = U \oplus W,$$

Then we decompose $W = W' \oplus V_\Delta$ by the diagonal map $V \to W$ and project r_V to $U \oplus W'$. This map can be regarded as $G = (Z_2)^{m+1}$-equivariant section of the bundle $\xi \oplus \eta$, where ξ arises from U and η arises from W'.

By Lemma 8 $e(\eta) \neq 0$ and the rest of the proof is similar the proof of Theorem 2. □

9. Proof of Theorems 5 and 6

Proof of Theorem 5. We prove this theorem just in the same way as we prove Theorem 1.

Let $r_i = p^{k_i}$ and put for all $i = 0, \dots, m$ $G_i = (Z_p)^{k_i}$. For each of the set S_i we take the simplicial complex of multicolored subsets K_i (as described above) and its r_i-fold deleted join $L_i = K^{*r_i}_{i\Delta}$ with appropriate G_i-action.

Denote $l = d - m$. The Grassmann variety will be oriented for odd p.

For any l-dimensional linear subspace $V \subseteq \mathbb{R}^d$ consider the natural piece-wise linear (or simply continuous for topological version) projection $K_i \to V$ and its r_i-fold join

$$h_{i,V} : L_i \to J^{r_i}_A(V),$$

The maps $h_{i,V}$ are equivariant under the actions of G_i. Take the cartesian product of these maps to get the $G = G_0 \times \cdots \times G_m$-equivariant map

$$h_{0,V} \times \cdots \times h_{m,V} : L_0 \times \cdots \times L_m \to J^{r_0}_A(V) \oplus \cdots \oplus J^{r_m}_A(V).$$

Now considering the dependence of this map on $V \in G^l_d$ we get an equivariant section of the bundle

$$\xi = J^{r_0}_A(\gamma) \oplus \cdots \oplus J^{r_m}_A(\gamma) \to L_0 \times \cdots \times L_m \times G^l_d.$$

Now consider the decomposition

$$\xi = D^{r_0}_A(\gamma) \oplus \cdots \oplus D^{r_m}_A(\gamma) \oplus (m+1)\gamma$$

and with the diagonal embedding $\gamma \to (m+1)\gamma$ we also have the decomposition $(m+1)\gamma = \gamma_\Delta \oplus \eta$. As in the previous proofs we identify $\eta = m\gamma$.

Thus denote

$$\zeta = D^{r_0}_A(\gamma) \oplus \cdots \oplus D^{r_m}_A(\gamma) \oplus \eta \to L_0 \times \cdots \times L_m \times G^l_d.$$

If the section constructed above in projection to ζ has a zero, we are done as in the proof of Theorem 1.

Finally we have to show that the equivariant Euler class

$$e(\zeta) \in H^*_G(L_0 \times \cdots \times L_m \times G^l_d, Z_p)$$

is nonzero. Note that by Lemma 8 we have $e(\eta) \neq 0 \in H^*(G^l_d, Z_p)$, and any embedding with fixed $V \in G^l_d$

$$g : L_0 \times \cdots \times L_m \to L_0 \times \cdots \times L_m \times G^l_d$$

induces the bundle

$$\beta = g^*(D^{r_0}_A(\gamma) \oplus \cdots \oplus D^{r_m}_A(\gamma)) \to L_0 \times \cdots \times L_m,$$

which is the cartesian product of G_i-bundles over L_i, arising from the representation of G_i in $D^{r_i}_A(V)$. Since $\dim D^{r_i}_A(V) = (r_i - 1)(d - m + 1)$, the construction of L_i and the conditions of the theorem imply that

$$r_i k - 1 \geq (r_i - 1)(d - m + 1)$$

and each of summand bundles has nonzero Euler class in $H^*(L_i, Z_p)$. Thus by Künneth's formula and the multiplicative rule for the Euler class

$$e(\beta) \neq 0 \in H^*_G(L_0 \times \cdots \times L_m, Z_p).$$

Using Künneth's formula for $(L_0 \times \cdots \times L_m)/G \times G^l_d$ and Lemma 7 we see that

$$e(D^{r_0}_A(\gamma) \oplus \cdots \oplus D^{r_m}_A(\gamma)) = e(\beta) \times 1 + \sum u \times v,$$

where $u \in H^*_G(L_0 \times \cdots \times L_m, Z_p)$, $v \in H^*(G^l_d, Z_p)$ and all v have $\dim v > 0$.

Now by the multiplicative rule for the Euler class

$$e(\zeta) = e(D^{r_0}_A(\gamma) \oplus \cdots \oplus D^{r_m}_A(\gamma))e(\eta) = e(\beta) \times e(\eta) + \sum u \times ve(\eta) \neq 0$$

in the cohomology $H^*_G(L_0 \times \cdots \times L_m \times G^l_d, Z_p)$. □

Proof of Theorem 6. Here we give only the differences from the previous proof.

Consider $r_i = t_i = 2$. In this case S_i has natural G_i-action ($G_i = Z_2$) that exchanges every two vertices with same color. This gives free G_i-action on K_i itself, so we can consider $L_i = K_i$. Also note that K_i is the boundary of k-dimensional cross-polytope, so K_i is a $k-1$-dimensional ($d-m$-dimensional) sphere.

In the case $r_i = 2$ instead of the map

$$h_{i,V} : L_i \to J^2_A(V)$$

we consider the map

$$h_{i,V} : K_i \to V \oplus V$$

given by $x \in K_i \mapsto \pi(x) \oplus \pi(-x)$, where by the minus sign we denote the action of Z_2. This is a Z_2-equivariant map when Z_2 permutes the summands of $V \oplus V$.

Instead of projection $J^2_A(V) \to D^2_A(V)$ we consider the projection $V \oplus V \to V$ given by $v_1 \oplus v_2 \mapsto v_1 - v_2$. The codomain space V here has antipodal Z_2 action and a point $x \in K_i$ maps to zero in V iff the projections of x and $-x$ to V coincide.

The space V has nonzero equivariant Euler class in $H^{d-m}_{Z_2}(\mathrm{pt}, Z_2)$ and in $H^{d-m}_{Z_2}(K_i, Z_2)$. The rest of the above proof works without change. □

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References

[1] Van Kampen R.E. Komplexe in euklidischen Räumen. // Abh. Math. Sem., Hamburg, 9, 1932, 72–78
[2] Flores A. Über n-dimensionale Komplexe, die im R_{2n+1} absolut selbstverschlungen sind. // Ergeb. Math. Kolloq., 6, 1932/1934, 4–7
[3] Tverberg H. A generalization of Radon's theorem. // J. London Math. Soc., 41, 1966, 123–128
[4] Dol'nikov V.L. Common transversals for families of sets in $\mathbb{R}^n$ and connections between theorems of Helly and Borsuk. (In Russian) // Doklady Akademii Nauk USSR., 297(4), 1987, 777-780
[5] Barany I. and Larman D.G. A colored version of Tverberg's theorem. // J. London Math. Soc., 45(2), 1992, 314–320
[6] Vrećica S.T., Živaljević R.T. The colored Tverberg's problem and complexes of injective functions. // Journal of Comb. Theory, Ser. A, 61(2), 1992, 309–318.
[7] Tverberg H., Vrećica S. On generalizations of Radon's theorem and the ham sandwich theorem. // Europ. J. Combinatorics, 14, 1993, 259–264
[8] Vrećica S.T., Živaljević R.T. New cases of the colored Tverberg theorem. // Jerusalem Combinatorics '93, ed. by H. Barcelo, G. Kalai, Contemporary Mathematics, Amer. Math. Soc., Providence, 1994, 325–334
[9] Volovikov A. Yu. On a topological generalization of the Tverberg theorem. // Mathematical Notes, 59(3), 1996, 324–326
[10] Živaljević R.T. User's guide to equivariant methods in combinatorics. // Publications de l'Institut Mathematique, 64(78), 1998, 107–132
[11] Živaljević R.T. The Tverberg-Vrećica problem and the combinatorial geometry on vector bundles. // Israel J. Math., 111, 1999, 53–76
[12] Vrećica S.T. Tverberg's conjecture. // Discrete and Computational Geometry, 29, 2003, 505–510
[13] Hsiang Wu Yi. Cohomology theory of topological transformation groups. Berlin-Heidelberg-New-York, Springer Verlag, 1975
[14] Hatcher A. Algebraic Topology. Cambridge University Press, 2002
[15] Matoušek J. Using the Borsuk-Ulam theorem. Berlin-Heidelberg, Springer Verlag, 2003

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