## 1. Segments in the line and axis-parallel rectangles

1.1 (Helly's theorem in the line). Prove that if every two segments in a finite family of segments in the line intersect then all the segments of the family have a common point.

1.2. Assume a finite family of segments in the line is given and no point in the line belongs to more than $k$ of the segments. Prove that the segments of the family can be colored in $k$ colors so that no two segments of the same color intersect.

1.3. Assume a finite family of segments in the line is given and of any $k+1$ of them some two intersect. Prove that there exists a set $X$ of $k$ points such that any segment of the family contains at least one point of $X$, that is *the family of segments is pierced by* $X$.

1.4. Assume a family of axis-parallel rectangles in the plane is given. Prove that if any two rectangles in the family intersect then the whole family has a common point.

1.5. Assume a family of axis-parallel unit squares in the plane is given, without $k+1$ pairwise disjoint. Prove that there exists a set $X$ of $2k-1$ points such that the family of squares in pierced by $X$.

1.6. Assume a family of axis-parallel squares (of arbitrary size) in the plane is given, without $k+1$ pairwise disjoint. Prove that there exists a set $X$ of $4k-3$ points such that the family of squares in pierced by $X$.

1.7. ** Assume a family of axis-parallel rectangles (of arbitrary size) in the plane is given, without $k+1$ pairwise disjoint. Find a way to produce a set $X$ of smallest possible size such that the family of rectangles in pierced by $X$.

1.8. Assume a family of axis-parallel rectangles in the plane is given and any two of them can be crossed by either a horizontal or a vertical line. Prove that there exists a pair of lines, one horizontal and the other vertical, that crosses all the rectangles in the family.

## 2. Parity, angle counting, and Euler's formula

2.1. Prove that two closed polygonal lines in the plane in general position intersect in an even number of points. What is *general position* in this case?

2.2 (Jordan's lemma for polygonal lines). Prove that a closed polygonal line in the plane without self-intersections partitions the plane in two connected components.

2.3 (General case of Jordan's lemma). * Prove that a closed continuous curve in the plane without self-intersections partitions the plane in two connected components.

2.4. A *(not necessarily convex) polygon* in the plane is a closed polygonal line without self-intersections together with the bounded connected component of the plane it makes by Jordan's theorem. Prove that if a polygon has at least 4 vertices can be cut in two parts by a segment between two of its vertices, and the segment will only touch the boundary of the polygon in its endpoints.

2.5. Prove that a polygon in the plane can be cut (partitioned) into triangles with vertices at the original vertices of the polygon.

2.6. Prove that a polygon with $n$ vertices (an *$n$-gon*) has the sum of inner angles equal to

$$\pi(n-2).$$

2.7. Assume $N$ points are given in the interior of a square. Then the pairs of points, including the vertices of the square, are connected by segments only intersecting each other in endpoints, until it is impossible to draw such segments any more. Find the number of the segments drawn this way, show that it only depends on $N$.

2.8 (Euler's formula). Prove that, for a connected graph drawn in the plane with edges only intersecting each other in their endpoint vertices, the formula holds:

$$V - E + F = 2,$$

where $V$ is the number of vertices of the graph, $E$ is the number of edges, and $F$ is the number of regions graph partitions the plane.

2.9. Let real numbers $D, A > 0$ be given. Prove that it is impossible to partition the plane into convex 7-gons, each having diameter at most $D$ and area at least $A$.

2.10. Prove that if one of the restrictions, on diameter or on the area, in the previous problem is dropped, the partition becomes possible.

2.11. Prove that if a square is partitioned into triangles, then some two of the triangles have a whole common side.

2.12. Prove that a convex $n$-gon cannot be partitioned into less than $n-2$ triangles.

2.13. Prove that a convex polytope in $\mathbb{R}^3$ with $n$ vertices cannot be partitioned into less than $n-3$ tetrahedra.

## 3. Point and line sets in the plane

**Definition 3.1.** A point set in the plane is in *general position* if no three of them belong to the same line. A set of (straight) lines in the plane is in *general position* if no three of them have a common point and no two are parallel.

3.2. Any $n$ lines in general position have $\frac{n(n-1)}{2}$ intersections points and partition the plane into $\frac{n(n+1)}{2} + 1$ parts.

3.3. Let $n$ lines in general position in the plane be given. We want to choose the direction on every line so that the following holds: If we go along any line in its direction and put numbers from 1 to $n-1$ on the intersection points then no two equal numbers (coming from the two lines) appear at the same point. For which numbers $n$ is it possible?

3.4. Let $n$ lines in general position partition the plane. Prove that it is possible to put the nonzero positive and negative integers not exceeding $n$ by absolute value in the parts so that the, for any line, the sum of the numbers on one side and on the other side of it equals zero. What happens if the lines are not in general position?

3.5 (Sylvester's problem). Prove that if a finite point set in the plane does not belong to a single line then there exists a line passing through precisely two points of the set.

3.6. Let a finite set of lines be given in the plane. Prove that either they all have a common point, or they all are parallel, or there exists a point belonging to precisely two of them.

3.7. Let a finite family of red and blue lines in the plane be given. Assume no two two given lines are parallel and through any intersection point of two lines of the same color there passes a line of the other color. Prove that all the given lines in fact have a common point.

3.8. Let $n$ points be given in the plane, not belonging to a single line. Assume we draw all possible lines through pairs of the given points. Prove that we have drawn at least $n$ lines.

3.9. * What version of Sylvester's problem makes sense for the points in $\mathbb{R}^3$?

## 4. Convex polygons and convex sets

4.1. Prove that a convex polygon $P$ can be partitioned into parallelograms if and only if it $P$ is centrally symmetric.

4.2. Prove that if a convex polygon $P$ is partitioned into parallelograms then at least one of the parallelograms has at least two sides on the boundary of $P$.

**Definition 4.3.** An *affine diameter* of a convex body $K$ is a segment $I \subset K$ such that $K$ contains no longer segment in the direction of $I$.

4.4. Prove that a segment $I$ is an affine diameter of $K \subset \mathbb{R}^2$ if and only if its endpoint are on the boundary of $K$ and $K$ is contained in a plank (an area between two parallel lines) with endpoint of $I$ on the boundary of the plank.

4.5. Prove that, for any convex body $K \subset \mathbb{R}^2$, there exists an affine diameter of in any prescribed direction, and there exists an affine diameter through any prescribed point in the plane. Are there sufficient conditions for the uniqueness?

4.6. Let $A_1A_2\dots A_n$ be a convex $n$-gon without parallel sides. For any side $A_iA_{i+1}$, (assuming $A_{n+1} = A_1$) find the farthest from the line $A_iA_{i+1}$ vertex $A_k$ and denote the triangle formed by the side and the vertex by $T_i$. Prove that the triangles $T_i$ ($i = 1, \dots, n$) cover the given $n$-gon.

4.7. Let $A_1A_2\dots A_n$ be a convex $n$-gon without parallel sides. For any side $A_iA_{i+1}$, (assuming $A_{n+1} = A_1$) find the farthest from the line $A_iA_{i+1}$ vertex $A_k$ and consider the sum of such angles $\angle A_iA_kA_{i+1}$. Prove that the sum is $\pi$.

4.8. Let $A_1A_2\dots A_n$ be a convex $n$-gon with all sides of equal length. Prove that there exists at least two $k = 1, \dots, n$ such that

$$\angle A_{k-1}A_kA_{k+1} \ge \angle A_{k-2}A_{k-1}A_k \quad \angle A_{k-1}A_kA_{k+1} \ge \angle A_kA_{k+1}A_{k+2},$$

where we put $A_{-1} = A_{n-1}$, $A_0 = A_n$, $A_{n+1} = A_1$, and $A_{n+2} = A_2$.

4.9. Let $A_1A_2\dots A_n$ be a convex $n$-gon with all angles equal. Prove that there exists at least two $k = 1, \dots, n$ such that

$$\angle A_{k-1}A_kA_{k+1} \ge \angle A_{k-2}A_{k-1}A_k \quad \angle A_{k-1}A_kA_{k+1} \ge \angle A_kA_{k+1}A_{k+2},$$

where we put $A_{-1} = A_{n-1}$, $A_0 = A_n$, $A_{n+1} = A_1$, and $A_{n+2} = A_2$.

## 5. Carathéodory's, Radon's, and Helly's theorems and their analogues

5.1 (Carathéodory's theorem). The *convex hull* of a set $X \subseteq \mathbb{R}^n$ is the set of all convex combinations of points in $X$ denoted by $\operatorname{conv} X$. Prove that

$$\operatorname{conv} X = \bigcup_{Y \subseteq X \ |Y| \le n+1} \operatorname{conv} Y.$$

5.2 (Fenchel's theorem). Assume in the previous theorem that the set $X$ is connected and prove that

$$\operatorname{conv} X = \bigcup_{Y \subseteq X \ |Y| \le n} \operatorname{conv} Y.$$

5.3 (Colorful Carathéodory's theorem by Bárány). Assume $X_0, \dots, X_n$ are subsets of $\mathbb{R}^n$ each containing the origin in the convex hull. Prove that there exists a system of representatives $x_0 \in X_0, x_1 \in X_1, \dots, x_n \in X_n$ such that

$$0 \in \operatorname{conv}\{x_0, x_1, \dots, x_n\}.$$

5.4. Assume that $X_1, \dots, X_n$ are connected subsets of $\mathbb{R}^n$ each containing the origin in the convex hull. Prove that there exists a system of representatives $x_1 \in X_1, \dots, x_n \in X_n$ such that

$$0 \in \operatorname{conv}\{x_1, x_1, \dots, x_n\}.$$

5.5 (Radon's theorem). Prove that any set $X$ of $n+2$ points in $\mathbb{R}^n$ can be partitioned into two nonempty sets $X = Y \sqcup Z$ so that

$$\operatorname{conv} Y \cap \operatorname{conv} Z \neq \emptyset.$$

5.6. $t \leq n+1$ $2t \geq n+2$. , $X$ $n+2$ $\mathbb{R}^n$ $X = Y \sqcup Z$ , $|Z| = t$

$$\operatorname{conv} Y \cap \operatorname{aff} Z \neq \emptyset.$$

5.7 (Helly's theorem). Let a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^n$ has the property that any $\leq n+1$ of the sets in the family have a common point. Prove that all the sets in $\mathcal{F}$ have a common point, that is $\bigcap \mathcal{F} \neq \emptyset$.. Is this true for infinite families?

5.8. Let $X$ be a compact subset of $\mathbb{R}^n$. Prove that if any $\leq n+1$ points of $X$ can be covered by a unit ball then the whole $X$ can be covered by a unit ball.

5.9 (Jung's theorem). Prove that any set $X \subset \mathbb{R}^n$ of diameter at most $\sqrt{2+2/n}$ can be covered by a unit ball.

5.10. Let $X$ be a compact subset of $\mathbb{R}^n$. Prove that if any $\leq n+1$ points of $X$ can be covered by a unit ball not containing the origin then the whole $X$ can be covered by a unit ball not containing the origin.

5.11 (The center point theorem). Let a finite set $X \subset \mathbb{R}^n$ of $N$ points be given. Prove that there exists a point $p \in \mathbb{R}^n$ such that, for any half-space $H \ni p$, $|X \cap H| \geq \frac{N}{n+1}$.

5.12 (The colorful Helly theorem). Let some $n+1$ nonempty finite families of convex sets $\mathcal{F}_0, \dots, \mathcal{F}_n$ have the property: Any system of representatives

$$C_0 \in \mathcal{F}_0, \dots, C_n \in \mathcal{F}_n$$

has a common point. Prove that at least one of the families $\mathcal{F}_i$ has a common point.

5.13 (Fractional Helly's theorem in the plane). Prove that for any $\alpha \in (0,1)$ one can find $\beta(\alpha) \in (0,1)$ satisfying the following condition: If in a family $\mathcal{F}$ of $n$ convex bodies in the plane at least $\alpha\binom{n}{3}$ triples have a common point then there exists a point in the plane belonging to at least $\beta(\alpha)n$ sets of $\mathcal{F}$. Prove that it is possible to choose $\beta(\alpha)$ so that $\lim_{\alpha\to 1-0} \beta(\alpha) = 1$.

5.14 (Tverberg's theorem). Prove that any set $X \subset \mathbb{R}^n$ of $(n+1)(r-1)+1$ points can be partitioned into $r$ nonempty parts, $X = X_1 \sqcup \dots \sqcup X_r$, so that

$$\operatorname{conv} X_1 \cap \dots \cap \operatorname{conv} X_r \neq \emptyset.$$

5.15. Let 3 red, 3 green, and 3 blue points be given in the plane. Prove that it is possible to partitions the points into 9 triples, each consisting of different colors, so that the three corresponding triangles have a common point.

5.16 (Colorful Tverberg's theorem by Bárány and Larman). ** Let $n$ red, $n$ green, and $n$ blue points be given in the plane. Prove that it is possible to partitions the points into $n$ triples, each consisting of different colors, so that the $n$ corresponding triangles have a common point.

## 6. Piercing and Helly-type theorems for algebraic sets

6.1 (Strong Helly property of linear subspaces). Let $V$ be a vector space of dimension $n$ and let $\mathcal{F}$ be a family of its linear subspaces. Prove that there exist at $\le n$ representatives $L_1, \dots, L_m \in \mathcal{F}$ ($m \le n$) such that

$$\bigcap_{i=1}^{m} L_i = \bigcap_{L \in \mathcal{F}} L := \bigcap \mathcal{F}.$$

6.2. Let $V$ be a vector space of dimension $n$, let $\mathcal{F}$ be a family of its linear subspaces, and let $S \subset V$ be an arbitrary subset. Prove that there exists $\le n$ representatives $L_1, \dots, L_m \in \mathcal{F}$ ($m \le n$) such that

$$\bigcap_{i=1}^{m} L_i \cap S = \bigcap_{L \in \mathcal{F}} L \cap S = (\bigcap \mathcal{F}) \cap S.$$

6.3. Let $S$ be an arbitrary set, let $V$ be a vector space (under point-wise addition and multiplication) of functions $S \to \mathbb{K}$ (for some field $\mathbb{K}$) such that $\dim V = n$. For any $f \in V$, denote the zero set by $Z_f = \{x \in S : f(x) = 0\}$. Let $\mathcal{F}$ be a family of subsets of $S$ of the form $Z_f$ for some $f \in V$. Prove that there exist $\le n$ sets $X_1, \dots, X_m \in \mathcal{F}$ ($m \le n$) such that

$$\bigcap_{i=1}^{m} X_i = \bigcap \mathcal{F}.$$

6.4 (Helly's theorem for finite sets). Let $S$ be arbitrary set and let $\mathcal{F}$ be a family of its subsets, every subset having at most $n$ elements. Prove that there exist at most $n+1$ subsets $X_1, \dots, X_m \in \mathcal{F}$ ($m \le n+1$) such that

$$\bigcap_{i=1}^{m} X_i = \bigcap \mathcal{F}.$$

6.5. Prove that if a family of circles in the plane has the property that each $\le 4$ circles in the family have a common point then all the circles in the family have a common point.

6.6. Prove that if a family of circles in the plane has the property that each $\le 3$ circles in the family have a common point, and the family consists of at least 5 distinct circles, then all the circles in the family have a common point.

6.7. Let a finite point set be given in the plane so that any $\le 6$ of the given points can be crossed be a pair of lines. Prove that all the given points can be crossed by a pair of lines.

6.8. Let several graphs of polynomials of degree $\le d$ be given in the plane. Prove that if any $\le d+2$ of the graphs have a common point then all the given graphs have a common point.

6.9. For a family of sets $\mathcal{F}$, a set $T$ is called a *$t$-transversal* if $|T| \le t$ and any $X \in \mathcal{F}$ intersects $T$. Let $V$ be a vector space of dimension $n$, and let $\mathcal{F}$ be a family of its linear subspaces. Prove that if any subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size $|\mathcal{G}| \le \binom{n+t-1}{t}$ has a $t$-transversal then the whole $\mathcal{F}$ has a $t$-transversal.

6.10. Let $S$ be an arbitrary set, let $V$ be a vector space of functions $S \to \mathbb{K}$ (for some $\mathbb{K}$) such that $\dim V = n$. For any $f \in V$ denote the zero set by $Z_f = \{x \in S : f(x) = 0\}$. Let $\mathcal{F}$ be a family of subsets of $S$ of the form $Z_f$ for some $f \in V$. Prove that if any subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size $|\mathcal{G}| \le \binom{n+t-1}{t}$ has a $t$-transversal then the whole $\mathcal{F}$ has a $t$-transversal.

6.11. Let $S$ be an arbitrary set and let $\mathcal{F}$ be a family of its subsets, each having at most $n$ elements. Prove that if any subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size $|\mathcal{G}| \le \binom{n+t}{t}$ has a $t$-transversal then the whole family $\mathcal{F}$ has a $t$-transversal.

6.12 (Linear Hall's theorem). Assume finite subsets $S_1, \dots, S_n$ of a vector space $V$ are given with the following property: For any set of indices $1 \le i_1 < i_2 < \dots < i_k \le n$ (including one index) the linear span

$$\langle S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k} \rangle$$

has dimension at least $k$. Prove that there exists a linearly independent system of representatives $s_i \in S_i$.

6.13 (Colorful Helly's theorem for vector spaces). Let a vector space $V$ has dimension $n$, let $\mathcal{F}_1, \mathcal{F}_1, \dots, \mathcal{F}_n$ be families of linear subspaces of $V$. Prove that for a family of representatives $L_1 \in \mathcal{F}_1, \dots, L_n \in \mathcal{F}_n$ and some $j$

$$L_1 \cap \dots \cap L_n \subseteq \bigcap \mathcal{F}_j.$$

6.14 (Colorful Helly's theorem for finite sets). Let $S$ be an arbitrary set and let $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n$ be families of its subsets. Assume that all subsets in the families consist of at most $n$ elements and, for any system of representatives $X_i \in \mathcal{F}_i$ $(0 \le i \le n)$,

$$\bigcap_{i=0}^{n} X_i \neq \emptyset.$$

Prove that for some $i$ the intersection $\bigcap \mathcal{F}_i$ is nonempty.

6.15. Assume a family $\mathcal{F}$ of (affine) lines is given in $\mathbb{R}^n$, every two having a common point. Prove that either all lines of $\mathcal{F}$ lie in a two-dimensional affine plane or all the lines of $\mathcal{F}$ have a common point.

6.16. Assume a family $\mathcal{F}$ of affine $k$-dimensional subspaces in $\mathbb{R}^n$ is given so that any $\le k+2$ of the subspaces have a common point. Prove that $\bigcap \mathcal{F} \neq \emptyset$.

6.17. Let a finite set of $\ge d+1$ points in the plane with pairwise distinct $x$ coordinates be given. Assume that any graph of a degree $\le d$ polynomial through any $d+1$ of the points passes through at least one more of the given points. Prove that all the given points belong to the graph of a polynomial of degree $\le d$.

6.18. Prove that, for any positive integer $k$, there exists a subset $X \subset \mathbb{R}^3$ of $2k+3$ points such that every plane through the origin has at least $k$ of the given points strictly on each of its sides.

6.19. Let $n$ red and $m$ blue lines in the plane be given and $nm$ points of the intersection of lines of different colors are marked. Assume a family of green lines crosses all the marked points but precisely one. Prove that the number of green lines is at least $n+m-2$.

6.20. For any two finite sets of reals, $A$ and $B$, put

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Prove that we have the inequality for the cardinalities of the sets

$$|A + B| \ge |A| + |B| - 1.$$

6.21. For two sets of residues modulo a prime $p$, $A$ and $B$, put

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Prove that we have the inequality for the cardinalities of the sets

$$|A + B| \ge \min\{|A| + |B| - 1, p\}.$$

## 7. Inequalities for volumes and integrals

7.1 (Brunn–Minkowski inequality in the line). Let $X, Y \subseteq \mathbb{R}^1$ be measurable sets, denote the Lebesgue measure by vol. Prove the inequality

$$\mathrm{vol}(X+Y) \geq \mathrm{vol}(X) + \mathrm{vol}(Y),$$

where $X + Y = \{x + y, x \in X,\ y \in Y\}$.

7.2 (Functional Brunn–Minkowski inequality in the line). Let $f, g, h : \mathbb{R} \to \mathbb{R}^+$ be measurable functions and let $t \in (0,1)$. Assume that for any $x, y \in \mathbb{R}$ we have

$$h((1-t)x + ty) \geq f(x)^{1-t} g(x)^t.$$

Prove that

$$\int_{-\infty}^{+\infty} h(x)\, dx \geq \left(\int_{-\infty}^{+\infty} f(x)\, dx\right)^{1-t} \cdot \left(\int_{-\infty}^{+\infty} g(x)\, dx\right)^{t}.$$

7.3 (Functional Brunn–Minkowski inequality). Prove that the previous inequality holds for functions of $n$ variables and their integrals over $\mathbb{R}^n$ by induction in $n$.

7.4 (Brunn–Minkowski inequality for volumes). Prove that for any two measurable $X, Y \subseteq \mathbb{R}^n$ and $t \in (0,1)$ we have

$$\mathrm{vol}((1-t)X + tY) \geq \mathrm{vol}(X)^{1-t} \cdot \mathrm{vol}(Y)^t.$$

Using scalings of $X$ and $Y$ and varying $t$ show that

$$\mathrm{vol}(X+Y)^{1/n} \geq \mathrm{vol}(X)^{1/n} + \mathrm{vol}(Y)^{1/n}.$$

7.5. Denote the $t$-neighborhood of $X$ by $U_t(X)$. Define the lower Minkowski surface area by

$$\underline{\mathrm{vol}}_{n-1}^{+} X = \liminf_{t \to +0} \frac{\mathrm{vol}\, U_t(X) - \mathrm{vol}\, X}{t}.$$

Prove that for $X \subset \mathbb{R}^n$ with boundary of class $C^2$ this coincides with the Riemannian $(n-1)$-dimensional volume of the boundary.

7.6 (Isoperimatric inequality for the Minkowski surface area). Prove that for fixed $\mathrm{vol}\, X$ the minimum of $\underline{\mathrm{vol}}_{n-1}^{+} X$ is attained when $X$ is a ball. Write down the corresponding estimate for the surface area through the volume.

7.7 (The Prékopa–Leindler inequality). Let a density function $f : \mathbb{R}^n \to (0, +\infty)$ be $k$-strongly logarithmically concave for some $k \geq 0$, that is

$$d^2(\log f) \leq -2k(dx_1^2 + \dots + dx_n^2)$$

in the sense of distributions. Prove that its "projection"

$$g(x_1, \dots, x_{n-1}) = \int_{-\infty}^{+\infty} f(x_1, \dots, x_{n-1}, x_n)\, dx_n$$

is also $k$-strongly logarithmically concave.

7.8. A Borel measure $\mu$ in $\mathbb{R}^n$ is *logarithmically concave* if

$$\mu((1-t)X + tY) \geq \mu(X)^{1-t} \cdot \mu(Y)^t$$

for any convex bodies $X$ and $Y$. Prove that for a measure with density this is equivalent to the logarithmical concavity of the density in the previous problem with $k = 0$.

7.9. * Prove that a finite logarithmically concave $\mu$ in $\mathbb{R}^n$ can be approximated by projections of the uniform measures on convex bodies.

7.10. Let $K$ and $L$ be centrally symmetric convex bodies. Prove that the maximum of the volume $\operatorname{vol} K \cap (L+t)$ over the shift vectors $t \in \mathbb{R}^n$ is attained at $t = 0$.

7.11 (The Rogers–Shephard inequality). Prove for a convex body $K \subset \mathbb{R}^n$ that

$$\operatorname{vol}(K - K) \le \binom{2n}{n} \operatorname{vol} K.$$

7.12 (The Grünbaum–Hammer theorem). Prove that in a convex body $K$ there exists a point $m \in K$ with the following property: Any half-space containing $m$ contains at least $1/e$ of the volume of $K$.

7.13 (The center point theorem for logarithmically concave measures). Prove that for a logarithmically concave measure $\mu$ in $\mathbb{R}^n$ there exists $m \in \mathbb{R}^n$ with the following property: For any half-space $H \ni m$,

$$\mu(H) \ge \frac{1}{e}\mu(\mathbb{R}^n).$$

7.14. * Prove that, for a pair of convex bodies $K$ and $L$, the expression

$$\operatorname{vol}(K + tL)$$

is a polynomial in $t \ge 0$. Prove that the coefficients of this polynomial are monotonic by inclusion functions of $K$ and $L$.

## 8. Polytopes

8.1. Prove that a bounded solution set of a system of linear inequalities in $\mathbb{R}^n$ is a *polytope*, that is a convex hull of a finite point set.

8.2. A partition of $\mathbb{R}^n$ is called *regular* if it is given by projecting the graph of a piece-wise linear convex function $f : \mathbb{R}^n \to \mathbb{R}$, that is consist of the maximal regions of linearity of such a function. Give examples of non-regular partitions of $\mathbb{R}^2$ into convex parts.

8.3 (Edelsbrunner's theorem). Prove that any regular partition of $\mathbb{R}^n$ is ordered with respect to any direction $\nu \in \mathbb{S}^{n-1}$, that is the parts can be ordered so that moving along any line in direction $\nu$ the parts are met without contradiction to their ordering.

8.4. * A *regular partition* of a set $X \subset \mathbb{R}^n$ is a restriction of a regular partition of $\mathbb{R}^n$ to $X$. Prove that if $B^n = P_1 \cup P_2 \cup \dots \cup P_N$ is a regular partition of a unit ball of some norm in $\mathbb{R}^n$ then every $P_i$ contains a ball (of that norm) of radius $r_i$ so that

$$r_1 + r_2 + \dots + r_N \ge 1.$$

8.5. * Prove that in dimension 2 in Problem 8.4 the regularity assumption is not needed.

8.6 (The Kadets theorem). ** Prove that in arbitrary dimension, for a Euclidean ball in Problem 8.4, the regularity assumption is not needed and instead of partitions one may consider coverings by convex sets.

8.7 (K. Bezdek's conjecture, an unsolved problem). ** Prove that in any dimension and any norm the regularity assumption in Problem 8.4 is not needed.

8.8 (Minkowski's theorem, the equality). Assume a polytope $P \subset \mathbb{R}^n$ of full dimension has facets $F_1, \dots, F_N$ with respective normals $\nu_1, \dots, \nu_N$ and areas $A_1, \dots, A_N$. Prove that

$$\nu_1 A_1 + \dots + \nu_N A_N = 0.$$

8.9 (Minkowski's theorem, existence and uniqueness). ** Prove that is the set of normals and areas satisfies

$$\nu_1 A_1 + \cdots + \nu_N A_N = 0,$$

and the normals span the whole $\mathbb{R}^n$ then there exists a polytope with such a set of normals and areas. Prove that the polytope is defined uniquely up to translations.

8.10. Let $\mathbb{R}^n$ be partitioned into translations of the same polytope $P$. Prove that $P$ is centrally symmetric.

8.11 (Voronoi's conjecture, an unsolved problem). **

Let $\mathbb{R}^n$ be partitioned into translations of the same polytope $P$. Prove that this such a partition is regular.

8.12 (The Bárány–Lovász theorem). ** A polytope in $\mathbb{R}^n$ is called *simple* if at any its vertex precisely $n$ edges meet. Prove that any centrally symmetric simple polytope in $\mathbb{R}^n$ has at least $2^n$ vertices.

8.13 (Stanley's theorem). ** Prove that any centrally symmetric simple polytope in $\mathbb{R}^n$ has at least $3^n$ faces of all dimensions, the polytope itself is considered a face.

8.14 (Unsolved problem). ** Prove that any centrally symmetric, not necessarily simple, polytope in $\mathbb{R}^n$ has at least $3^n$ faces of all dimensions, the polytope itself is considered a face.

## 9. Integer points

9.1. Prove that a convex 1000000-gon with integer vertices (both coordinates integer) has a side of length at least 550.

9.2. Prove that is a polygon with integer vertices has all side lengths equal then it has an even number of vertices.

9.3. Prove there does not exist regular $n$-gons with all vertices integer for $n = 3, 5, 6$.

9.4. Prove there does not exist regular $n$-gons with all vertices integer for $n \geq 7$.

9.5. A finite set of points in the plane is given. For every three of them there exists an orthogonal system of coordinates such that the three points have integer coordinates. Prove that there exists an orthogonal system of coordinates such that all the given points have integer coordinates.

9.6. Let a polytope $P$ in $\mathbb{R}^n$ has all vertices integer and the number of vertices at least $2^n + 1$. Prove that $P$ contains another integer point other than its vertices.

9.7 (Integer Helly's theorem). Let a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^n$ has the property: Any $2^n$ or less sets in the family have a common integer point. Prove that all the sets of $\mathcal{F}$ have a common integer point ($\mathbb{Z}^n \cap \bigcap \mathcal{F} \neq \emptyset$.).

9.8 (Minkowski's theorem). Let $K$ be a centrally symmetric convex body in $\mathbb{R}^n$ of volume at least $2^n$. Prove that $K$ contains an integer point other than the origin.

9.9. ** Let $K$ be a convex body in the place containing the origin in its interior and let $K^\circ$ be its polar body. Assume that $\operatorname{vol} K^\circ \leq 3/2$. Prove that $K$ contains an integer point other than the origin.

9.10. ** (Unsolved problem) Let $K$ be a convex body in $\mathbb{R}^n$ containing the origin in its interior. Let $\operatorname{vol} K^\circ \leq \frac{n+1}{n!}$ for the polar body. Prove that $K$ contains an integer point other than the origin.

## 10. Ramsey-type theorems

10.1. Prove that among $\binom{2k-4}{k-2}+1$ points of general positions in the plane one can choose $k$ points making a convex polygon.

10.2. Prove that in a set of $2^n$ points in the plane one can choose three points making a triangle with one angle at least $\left(1-\frac{1}{n}\right)\pi$.

10.3. Prove that there exists an arbitrarily large general position point set in the plane, such that it does not contain a set of seven points making a convex polygon without other points of this set inside it.

## 11. Delaunay triangulations and Voronoi partitions

**Definition 11.1.** A not necessarily convex polygon $F$ is triangulated if it is partitioned into triangles so that every two triangles either do not intersect, or intersect in a single vertex, or intersect in a whole side of both.

11.2. Prove that the convex hull of a finite point set $X$, not lying on a single line, can be triangulated so that the set of vertices of the triangles of the triangulation coincide with $X$.

**Definition 11.3.** Consider triangulations of the convex hull of a finite point set $X$ having precisely $X$ as the set of vertices of the triangles. Let us define a certain class of such triangulations. A *Delaunay triangulation* is such a triangulation with the additional property that the outscribed circle of any triangle of the triangulation has no points of $X$ in its interior.

11.4. Prove that for any $X$, not lying of a single line, there exists a Delaunay triangulation with vertices at $X$.

11.5. Prove that if $X$ is in general position and no 4 of its points lie on a single circle then two points $A$ and $B$ of $X$ are connected by an edge in the Delaunay triangulation if and only if there exists a ball $K$ such that $K\cap X=\{A,B\}$. In this case the Delaunay triangulation is unique.

11.6. Prove that any set of $n$ points in the plane can be colored in at most $10\ln n$ colors so that any ball intersecting the points contains precisely one point of some color.

**Definition 11.7.** The *Voronoi partition* for a finite point set $X$ is the partitions of the plane into sets $\{V_x\}_{x\in X}$ such that

$$V_x=\{p:\forall x'\in X\ |p-x|\le|p-x'|\}.$$

This definition also works in higher dimension.

11.8. Assume $X$ is a general position point set without 4 points on a single circle. What is the relation bewteen the Voronoi partions of $X$ and the Delaunay triangulation with vertices at $X$?

11.9. Assume a finite set of disjoint balls $K_1,K_2,\dots,K_n$ is given in the plane. Prove that the plane can be partitioned into convex parts $C_1,C_2,\dots,C_n$ so that for any $i=1,2,\dots,n$ we have $K_i\subseteq C_i$.

11.10 (Edelsbrunner's theorem). * Consider a union of balls $B_{c_i}(R_i)$ in $\mathbb{R}^n$. For any such ball, define a function

$$f_i(x)=R_i^2-|x-c_i|^2,$$

and a part

$$P_i=\{x\in\bigcup_i B_{c_i}(R_i):\forall j\ f_i(x)\ge f_j(x)\}.$$

Prove that the inclusion-exclusion formula can be simplified to give

$$\operatorname{vol} \bigcup_i B_{c_i}(R_i) = \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{P_{i_1}\cap\dots\cap P_{i_k}\neq\emptyset} \operatorname{vol}\left(B_{c_{i_1}}(R_{i_1}) \cap \dots \cap B_{c_{i_k}}(R_{i_k})\right).$$

## 12. Sets of vectors

12.1. Assume the points $A_1, A_2, \dots, A_N$ and $B_1, B_2, \dots, B_N$ are given in $\mathbb{R}^n$. Prove that the points $B_i$ can be reordered so that for any $i \neq j$ we will have (for the inner product of vectors)

$$\overline{A_iA_j} \cdot \overline{B_iB_j} \geq 0.$$

12.2. Assume $N$ vectors $\overline{a}_1, \overline{a}_2, \dots, \overline{a}_N$ are given in the plane with all lengths at most 1. Prove that it is possible to put $+$ and $-$ in place of $*$ in the expression

$$\overline{a}_1 * \overline{a}_2 * \dots * \overline{a}_N,$$

so that the result will have length at most $\sqrt{2}$.

12.3. * Assume $N$ vectors $\overline{a}_1, \overline{a}_2, \dots, \overline{a}_N$ are given in $\mathbb{R}^n$ with all lengths at most 1. Prove that it is possible to put $+$ and $-$ in place of $*$ in the expression

$$\overline{a}_1 * \overline{a}_2 * \dots * \overline{a}_N,$$

so that the result will have length at most $\sqrt{n}$.

## 13. Coverings and packings

**Definition 13.1.** For a closed convex set $X$, the *width in direction* $\ell$ is the length of the projection of $X$ to line $\ell$. The minimal width over all possible directions is usually called just *width*.

13.2. Prove that, for two convex bodies $X$ and $Y$ the following statements are equivalent:
1) Width of $X$ in any direction does not exceed the width of in the same direction $Y$;
2) $X + (-X) \subseteq Y + (-Y)$ (in the sense of Minkowski sum).

13.3. Prove that in any convex body $K \subset \mathbb{R}^n$ there exists a point $O$ such that any chord $AB \ni O$ (that is a segment with endpoint on $\partial K$) is split by $O$ with ratio $1 : n \leq \alpha \leq n : 1$.

13.4. Let two convex bodies $K$ and $L$ be given in the plane and the width of $K$ in any direction is at most $1/2$ of the width of $L$ in the same direction. Prove that it is possible to put $K$ into $L$ with a translation.

13.5. Let a finite point set $X$ and a regular triangle $T$ be given in the plane. Assume any two points of $X$ can be covered by a translate of $T$. Prove that the whole set $X$ can be covered by three translates of $T$.

13.6. Let a finite point set $X$ and a regular triangle $T$ be given in the plane. Assume any $\leq 9$ points of $X$ can be covered by two translates of $T$. Prove that the whole set $X$ can be covered by two translates of $T$.

13.7. Prove that any convex body $K \subset \mathbb{R}^n$ contains a translate of its homothet $-\frac{1}{n}K$.

13.8. Let a tetrahedron $T$ be contained in a ball of diameter 1. Prove that its width in some direction is at most $1/\sqrt{3}$.

13.9 (Moese's theorem). Prove that the planar ball of diameter 1 cannot be covered by planks of total width less than 1. A *plank* is a region between two parallel lines.

13.10 (The Goodman–Goodman theorem). * Let a family of balls (in arbitrary norm) in $\mathbb{R}^n$ has the following *non-separability property*: There exist no hyperplane that does not intersect any of the balls and has some balls on both sides of it. Prove that a non-separable set of balls of radii $R_1, R_2, \dots, R_N$ can be covered by one ball of radius $R_1 + R_2 + \dots + R_N$.

13.11 (The Kuperberg–Kuperberg theorem). * Prove that, for any convex body $K \subset \mathbb{R}^2$, its translates, together with the translates of $-K$, can be packed with density (the fraction of the covered area in arbitrarily big balls) at least $\sqrt{3}/2$.

13.12. Prove that a connected graph in $\mathbb{R}^n$ drawn by segments of total length 2 in some norm can be covered by a ball (in this norm) of radius 1.

13.13. Prove that a closed curve in $\mathbb{R}^n$ of length 4 in some norm can be covered by a ball (in this norm) of radius 1.

13.14. Prove that if we have a closed curve on the boundary of the unit Euclidean ball $B \subseteq \mathbb{R}^n$ of length less than $2\pi$ then it is possible to cover it with a ball of radius less than 1.